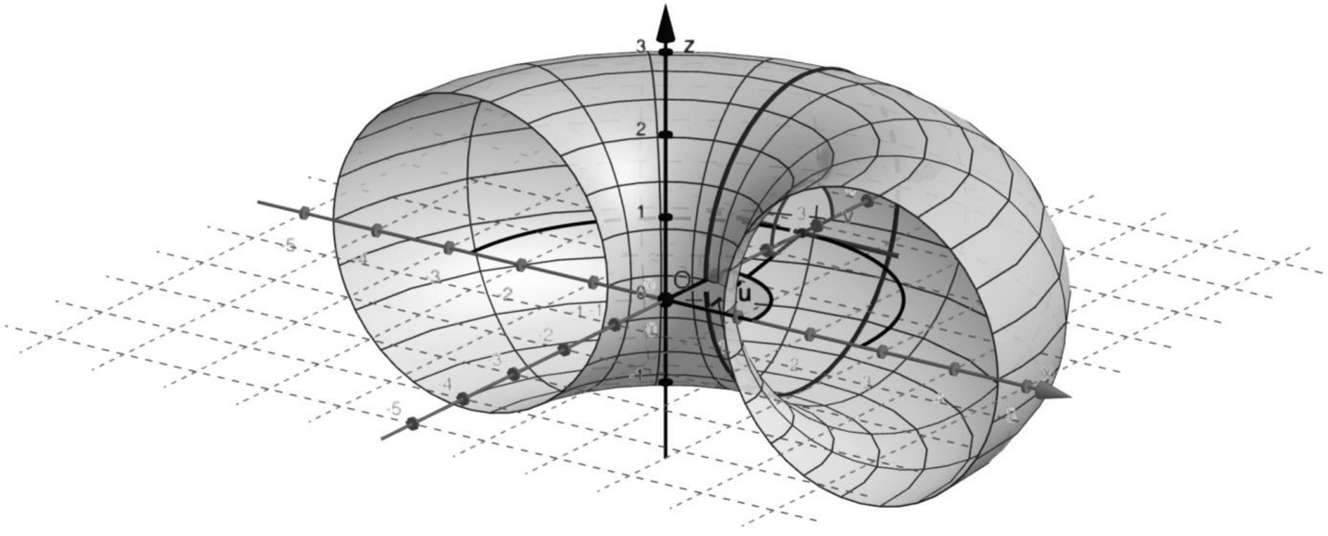
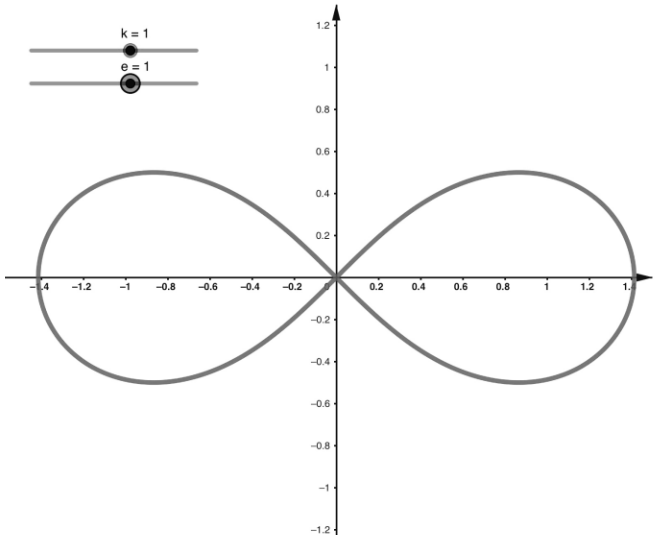
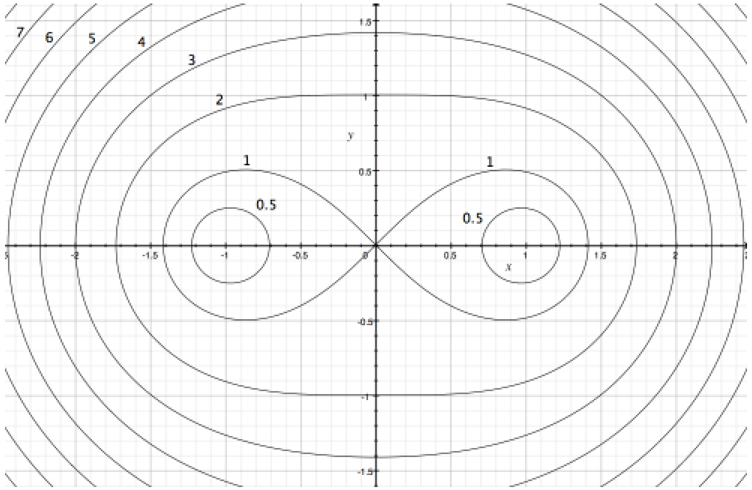
Untersuchung mathematischer Kurven

Sf48







Inhalt

- Einführung

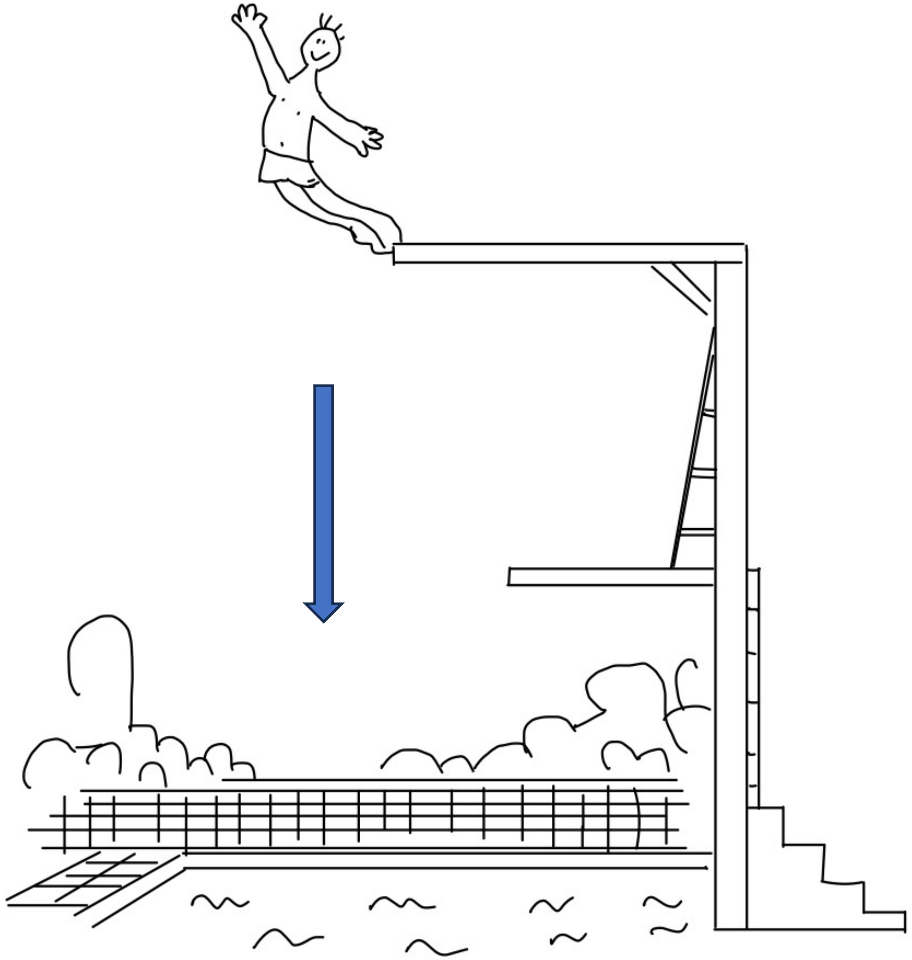
- Untersuchung von Kurven

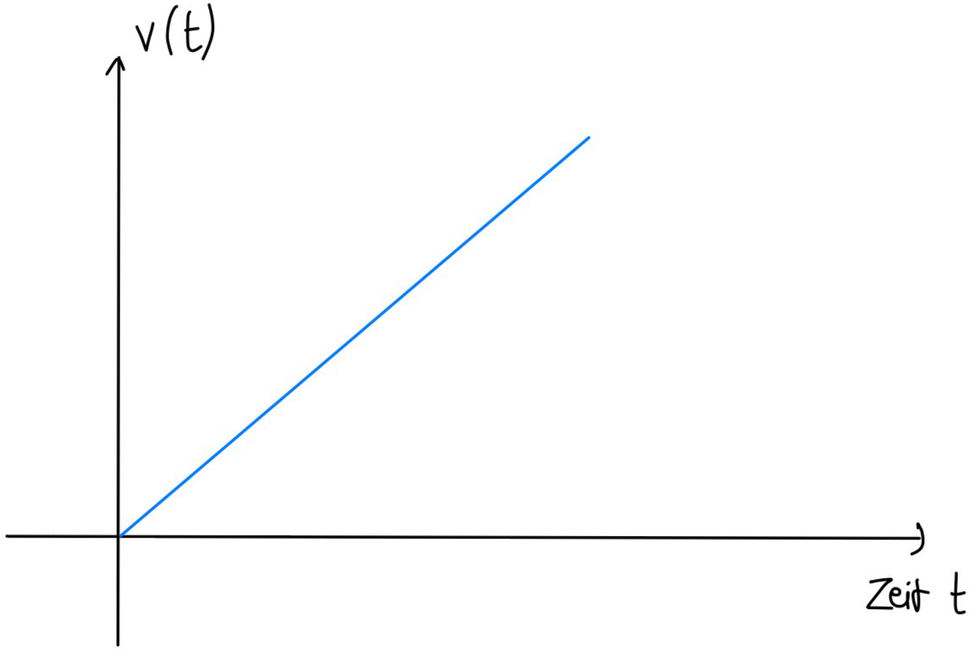
➢ Lemniskate

➢ Cassinische Kurve

- Höhere Kurven im Mathematikunterricht

Kurven in Natur und Technik





Kurvenbegriff

Heron von Alexandria
(100 n. Chr.)

Bahn eines Punktes

René Descartes
(1596-1650)

Algebraische und mechanische Kurven

Analytische Definition nach Camille Jordan

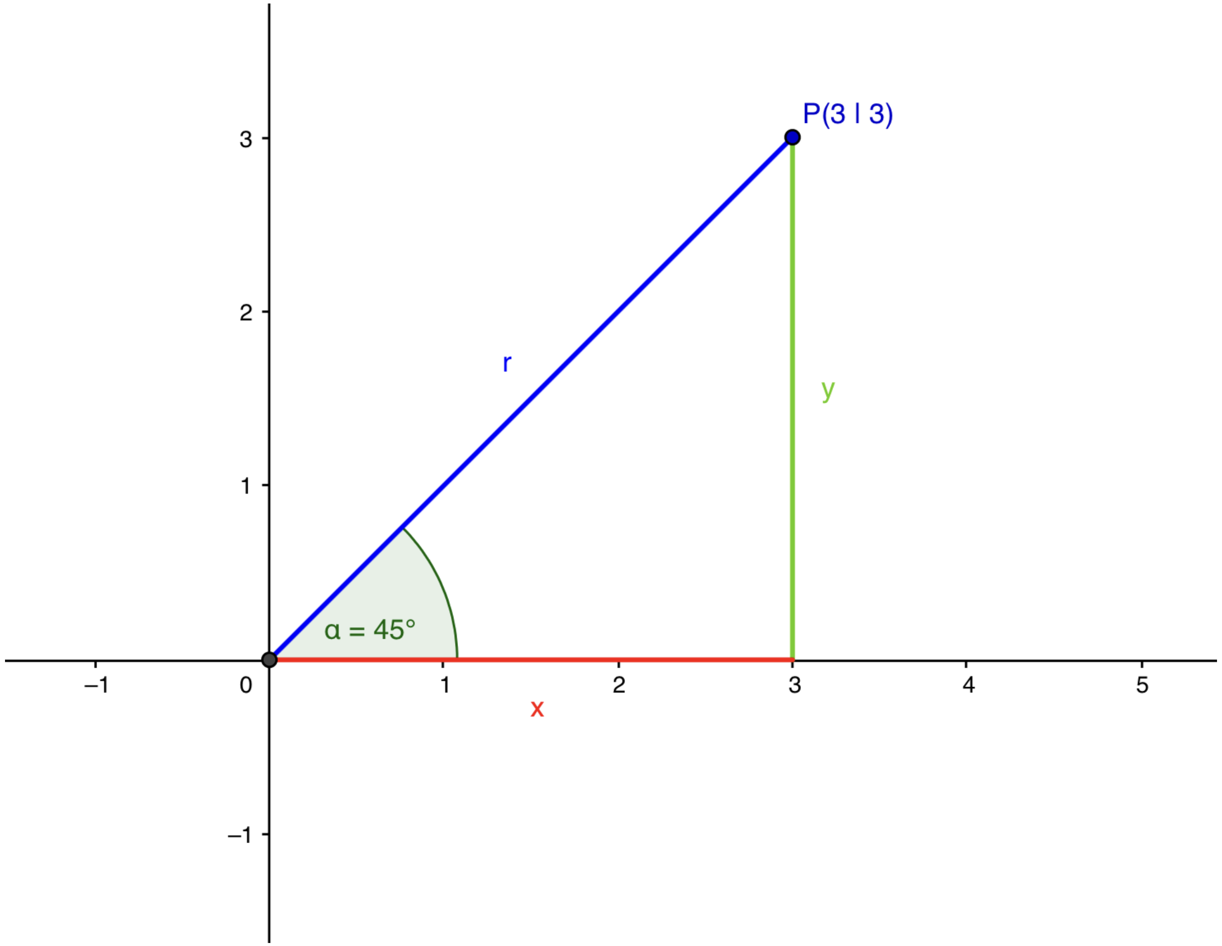
- Menge der Punkte $\boldsymbol{P = (x, y)}$

- Darstellung durch Parameter mit:

➢ $\boldsymbol{x = f(t)\ \ und\ \ y = g(t)}$

1838 - 1922

Polarkoordinaten



$$y = r \cdot \sin(\varphi)$$

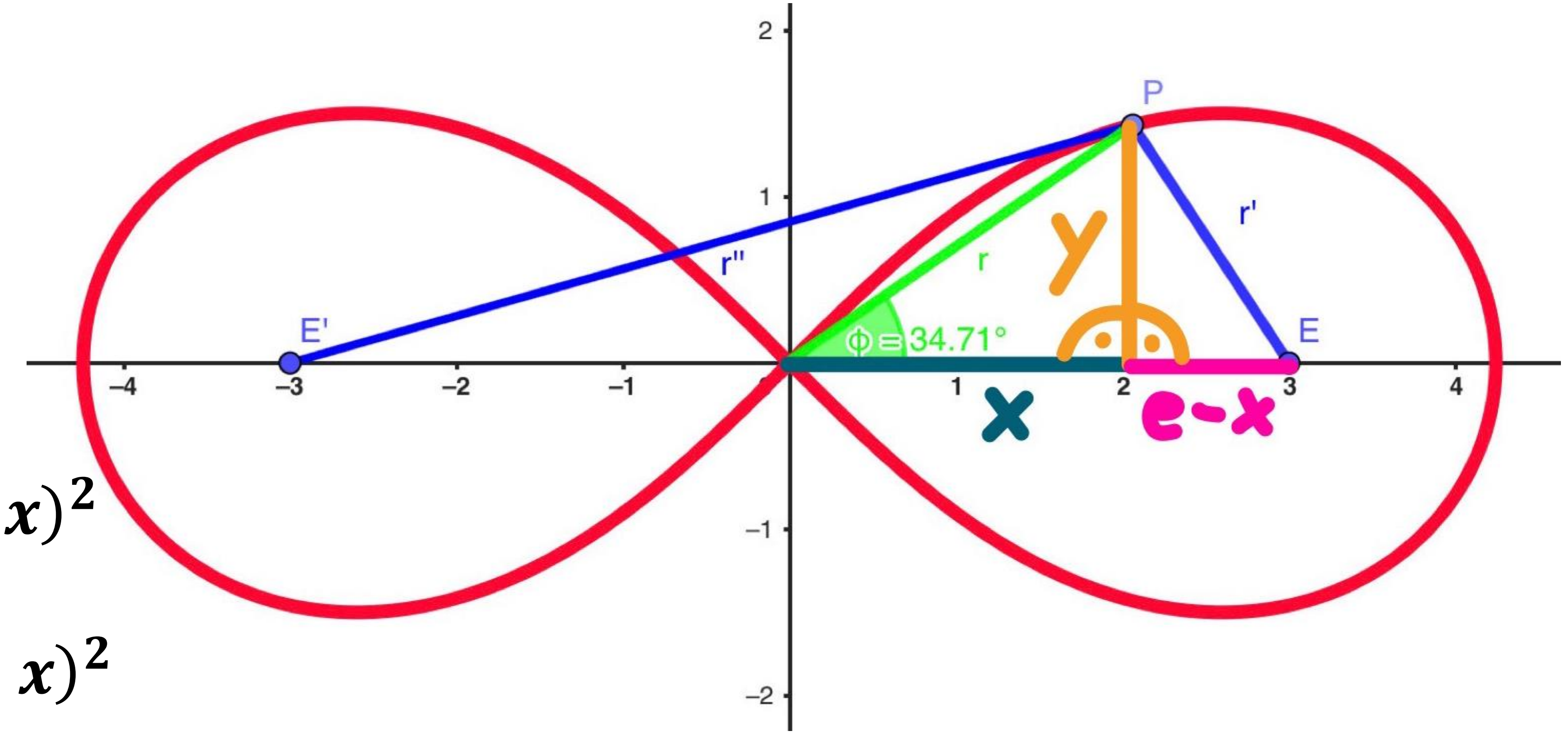
$$r^2 = x^2 + y^2$$

$$x = r \cdot cos\,(\varphi)$$

Untersuchung der Lemniskate

Herleitung der Gleichung

$$e^4 = (r')^2 \cdot (r'')^2$$



$$(r')^2 = y^2 + (e - x)^2$$

$$(r'')^2 = y^2 + (e + x)^2$$

Herleitung: Implizite kartesische Gleichung

$$e^4 = y^2(e-x)^2 + y^4 + y^2(e+x)^2 +$$

$$y^2(e+x)^2(e-x)^2$$

Herleitung: Implizite kartesische Gleichung

$$\cancel{e^4} = y^2e^2 - \cancel{y^2 2ex} + x^2y^2 + y^4 + y^2e^2 + \cancel{y^2 2ex} +$$

$$y^2x^2 + \cancel{e^4} + \cancel{2e^3x} + \cancel{e^2x^2} - \cancel{2e^3x} - \cancel{2e^2x^2} - \cancel{2ex^3} +$$

$$\cancel{e^2x^2} + \cancel{2ex^3} + x^4$$

Herleitung: Implizite kartesische Gleichung

$$0 = 2y^2e^2 + 2x^2y^2 + y^4 - 2e^2x^2 + x^4$$

$$-x^4 - 2x^2y^2 - y^4 = 2e^2(x^2 - y^2)$$

Implizite kartesische Gleichung

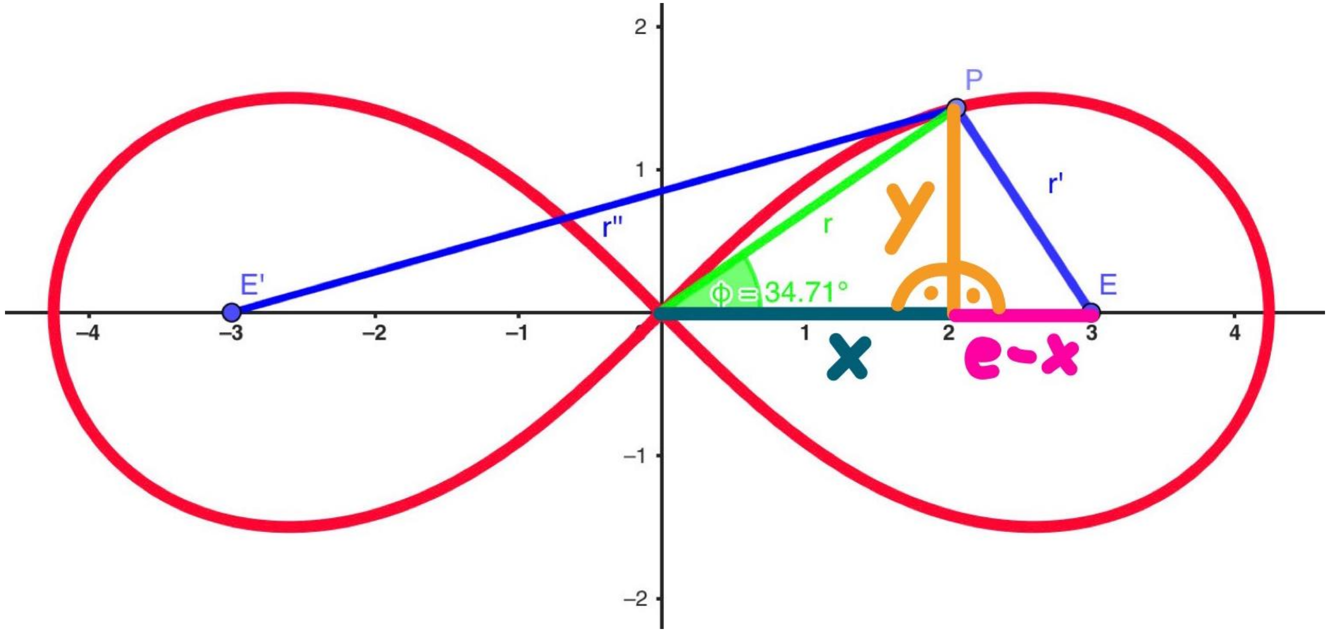
$$(x^2 + y^2)^2 = 2e^2(x^2 - y^2)$$

Herleitung der Gleichung

$$e^4 = (r')^2 \cdot (r'')^2$$

$$(r')^2 = y^2 + (e - x)^2$$

$$(r'')^2 = y^2 + (e + x)^2$$



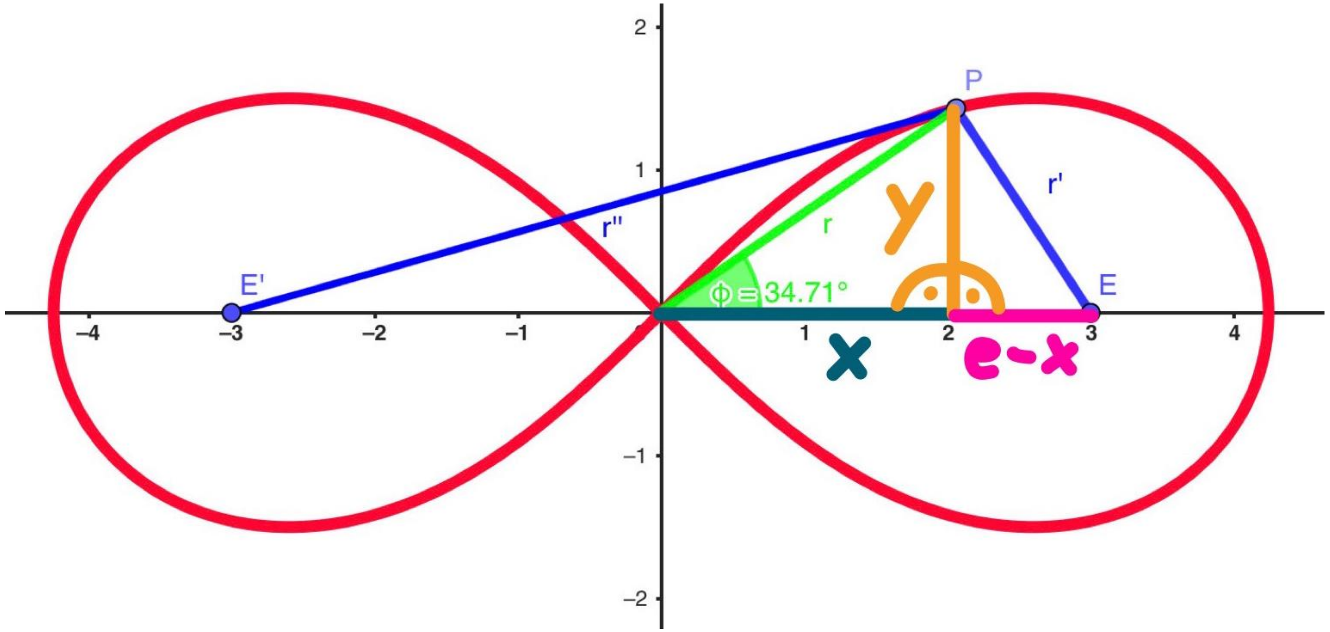
Herleitung der Gleichung in Polarkoordinaten

$$(r')^2 = \underline{y^2 + x^2} + e^2 - 2e\underline{x}$$

$$(r')^2 = \underline{r^2} + e^2 - 2e \cdot \underline{r \cdot \cos(\varphi)}$$

$$(r'')^2 = \underline{y^2 + x^2} + e^2 + 2e\underline{x}$$

$$(r'')^2 = \underline{r^2} + e^2 + 2e \cdot \underline{r \cdot \cos(\varphi)}$$



Herleitung der Gleichung in Polarkoordinaten

$$\cancel{e^4} = r^4 + r^2e^2 + \cancel{2er^3cos\varphi} + r^2e^2 + \cancel{e^4} +$$

$$\cancel{2e^3r \cdot cos\varphi} - \cancel{2er^3cos\varphi} - \cancel{2e^3r \cdot cos\varphi} -$$

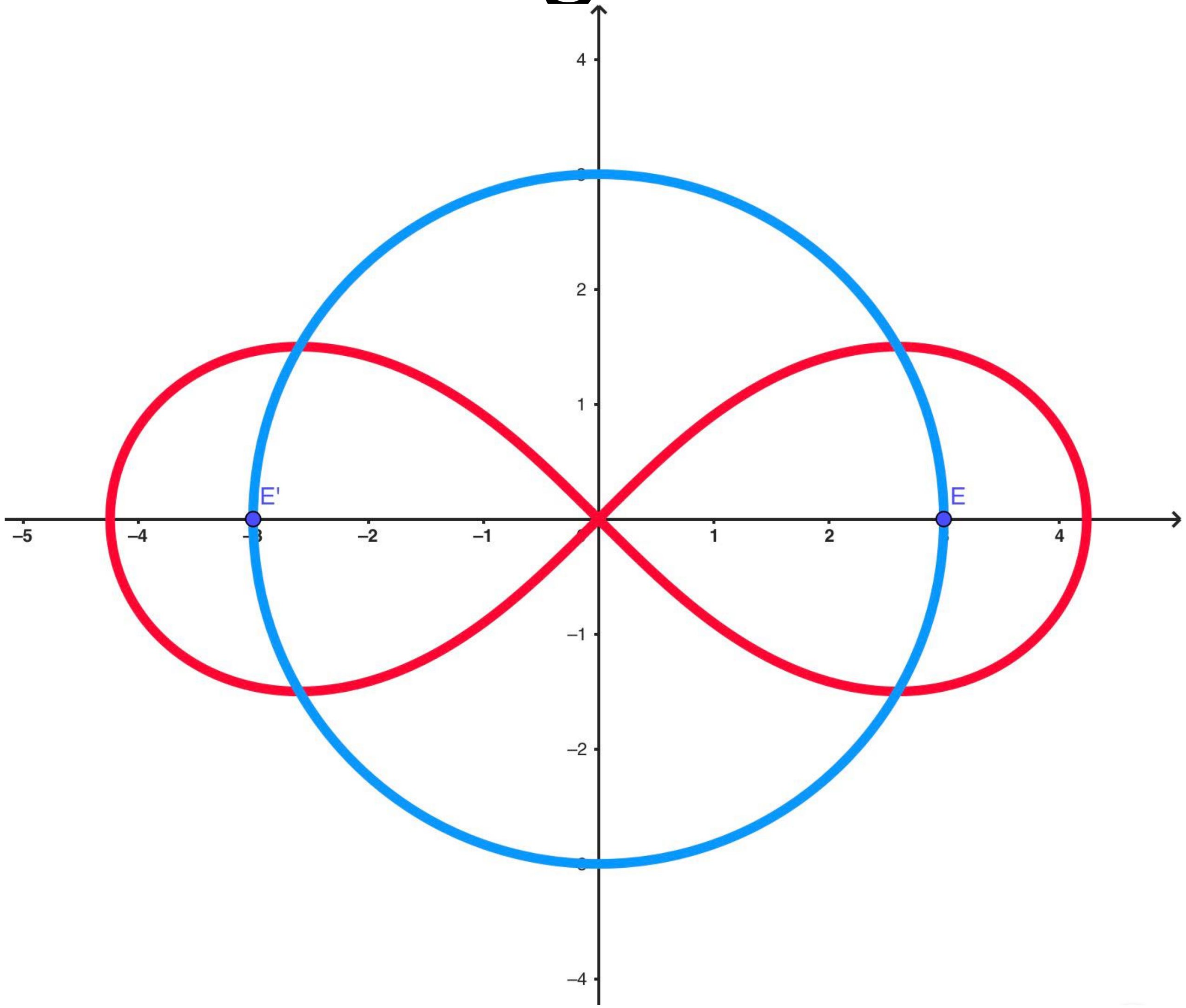
$$4e^2r^2(cos\varphi)^2$$

Gleichung in Polarkoordinaten

$$0 = r^4 + \ 2r^2e^2 - 4e^2r^2(cos\varphi)^2$$

$$\boldsymbol{r^2 = 2e^2 \cdot \cos(2\varphi)}$$

Ableitung



$$\left((x^2+y^2)^2\right)' = (2e^2(x^2-y^2))'$$

$$(x^4+2x^2y^2+y^4)' = (2e^2x^2-2e^2y^2)'$$

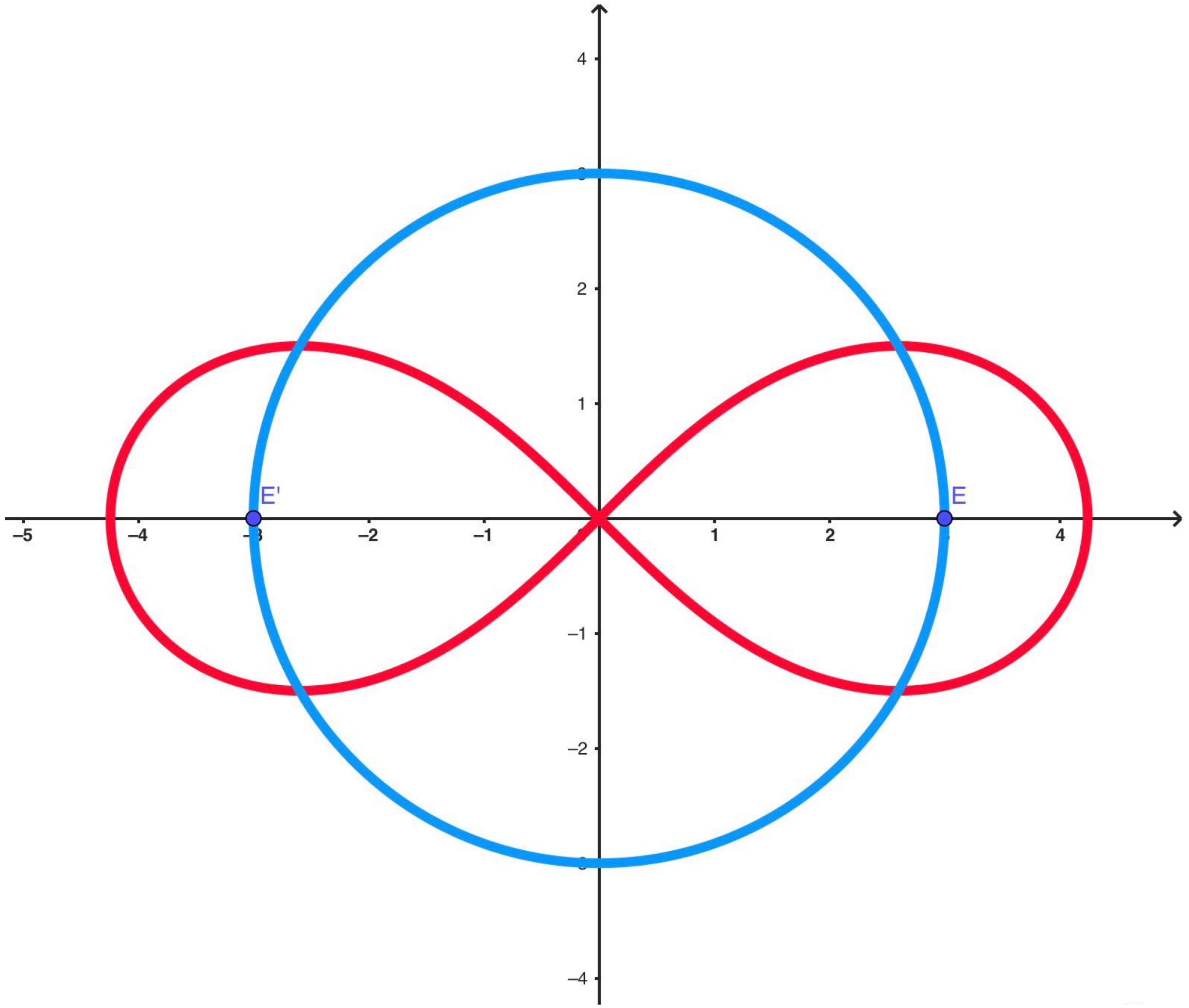
$$(x^4+2x^2y^2+y^4-2e^2x^2+2e^2y^2)' = 0$$

Ableitung

$$(x^4 + 2x^2y^2 + y^4 - 2e^2x^2 + 2e^2y^2)' = 0$$

$$4x^3 + 4xy^2 + 4x^2yy' + 4y^3y' - 4e^2x + 4^2yy' = 0$$

Ableitung

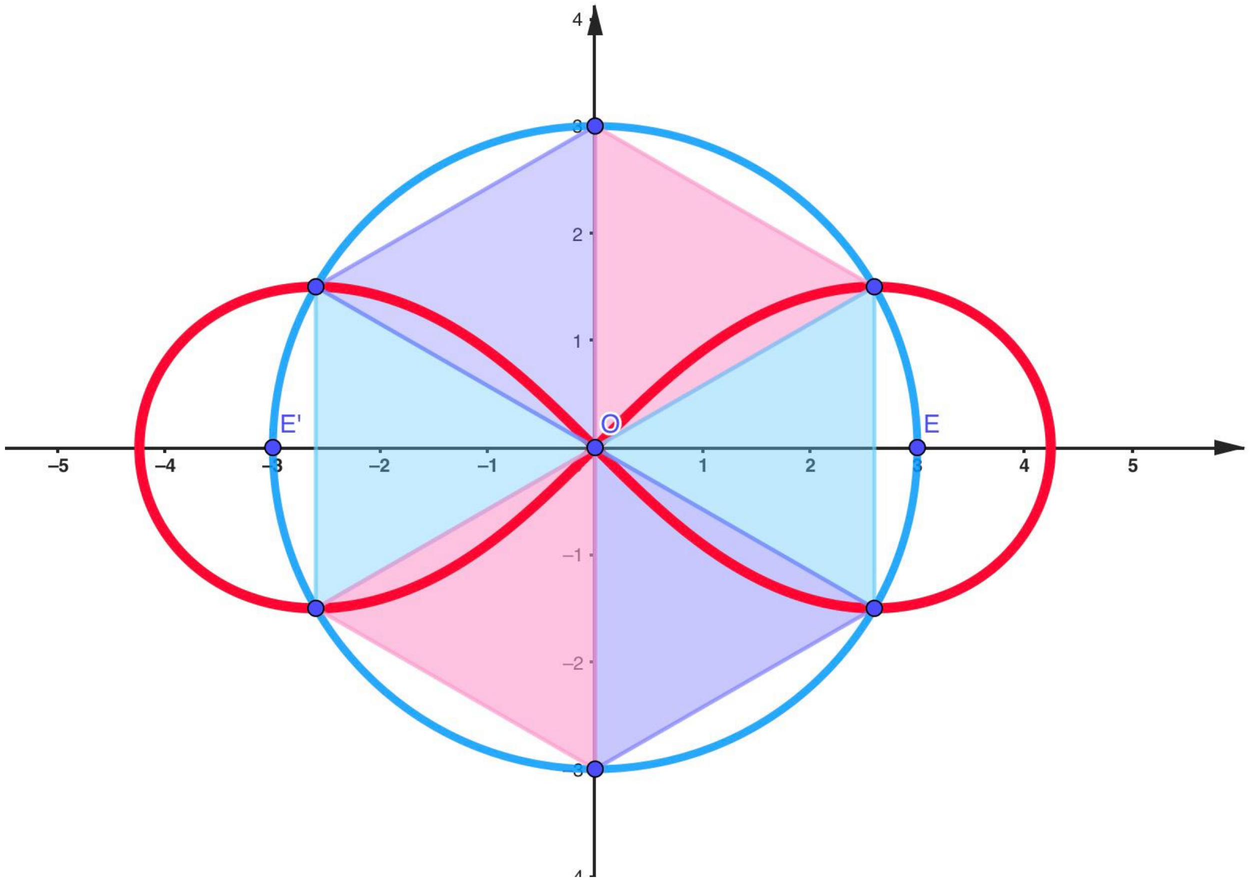


$$4x^3 + 4xy^2 + 4\cancel{x^2}yy' + 4\cancel{y^3}y' - 4e^2x + 4\cancel{e^2}yy' = 0$$

$$0 = 4x^3 + 4xy^2 - 4e^2x$$

$$x^2 + y^2 = e^2$$

Extrema



$$\left((e^2 - y^2) + y^2\right)^2 = 2e^2((e^2 - y^2) - y^2)$$

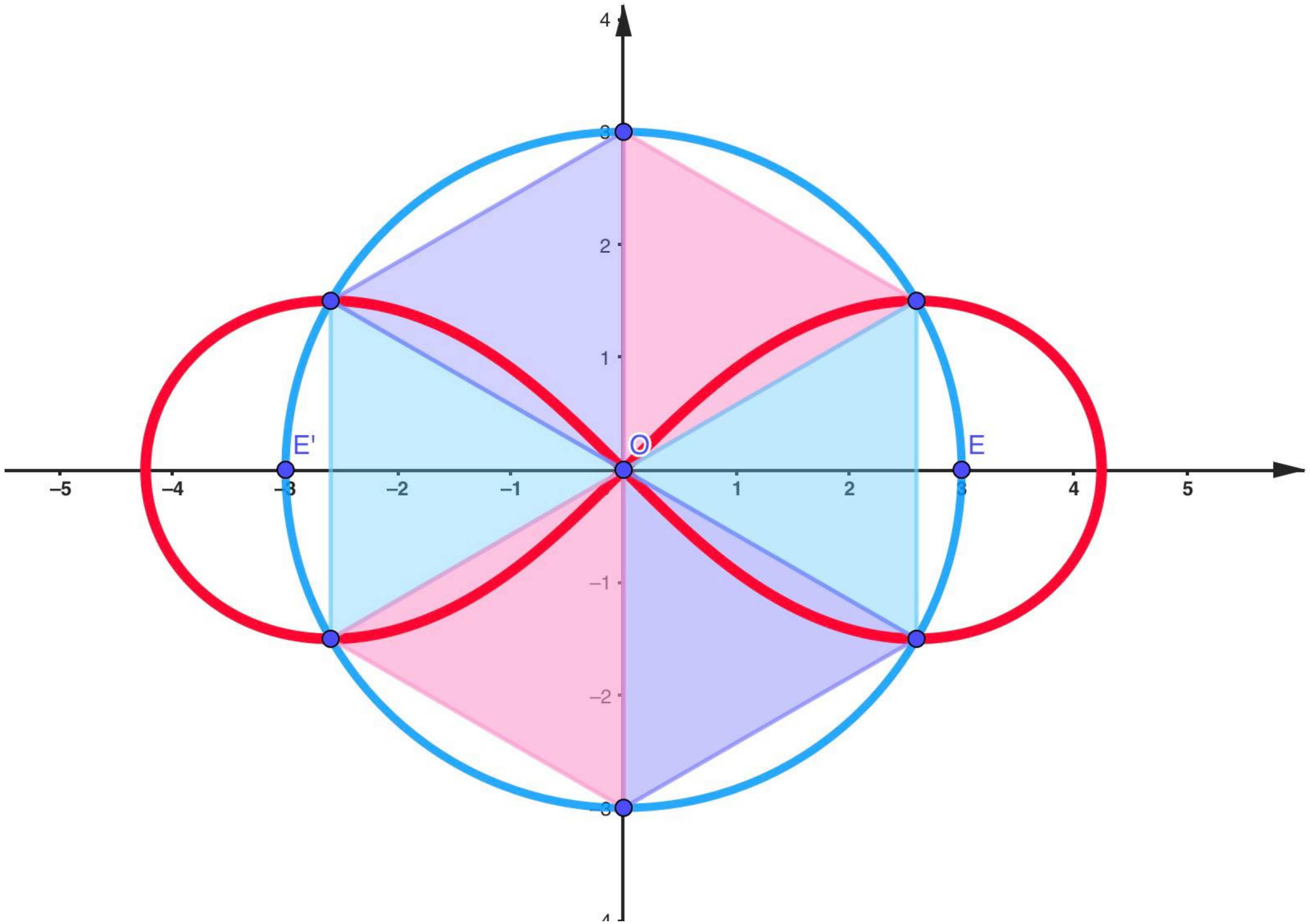
$$e^4 = 2e^4 - 4e^2y^2$$

$$4e^2y^2 = e^4$$

$$y^2 = \frac{e^2}{4}$$

$$y = \pm\frac{e}{2}$$

Extrema



$$0 = 4x^3 + 4x\left(\frac{e}{2}\right)^2 - 4e^{2x}$$

$$0 = 4x^3 + 4x\frac{e^2}{4} - 4e^2x$$

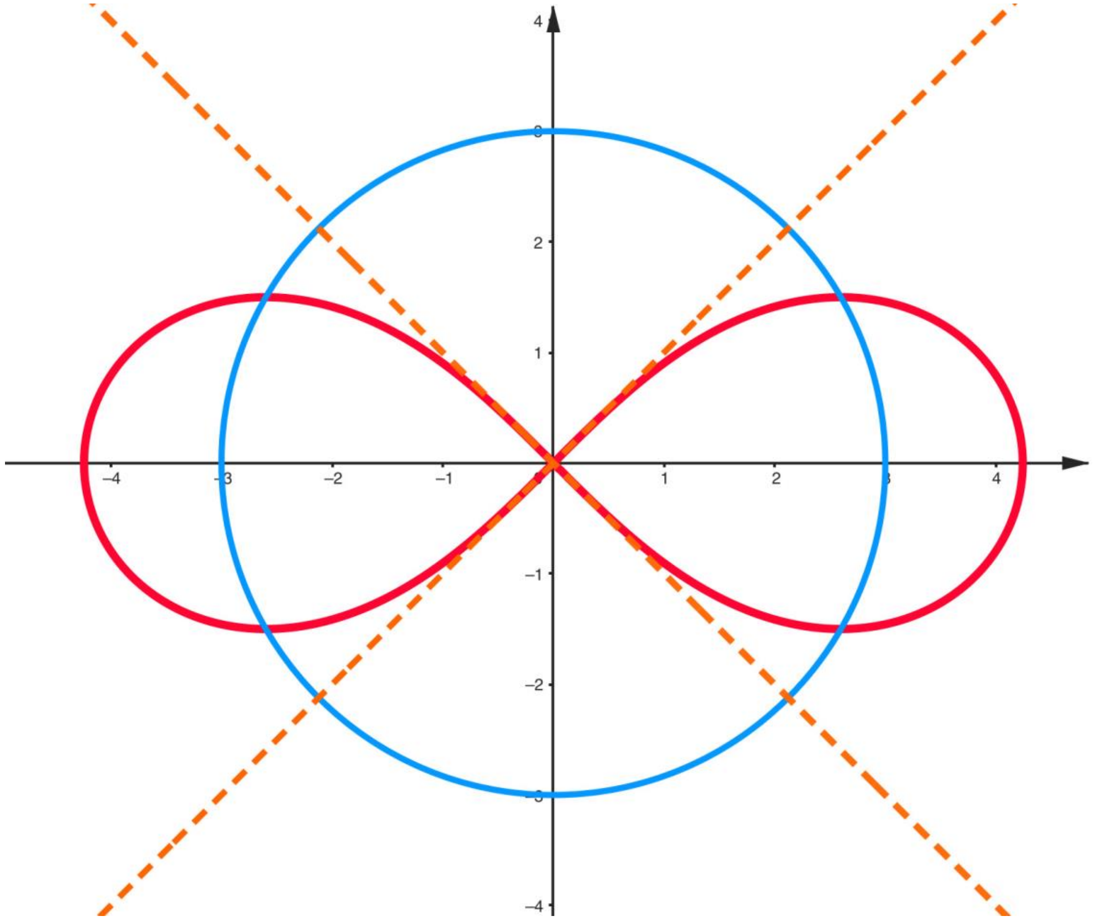
$$0 = 4x^3 + xe^2 - 4e^{2x}$$

$$0 = 4x^3 - 3e^{2x}$$

$$x = \pm\frac{\sqrt{3}}{2}e$$

$$EP\ \left(\pm\frac{\sqrt{3}}{2}e, \pm\frac{1}{2}e\right)$$

Ursprungstangenten



$$k\colon\ x^2 + y^2 = e^2 \qquad\qquad g\colon\ x = y$$

$$x^2 + x^2 = e^2$$

$$2x^2 = e^2$$

$$x = \frac{1}{2}\sqrt{2}e$$

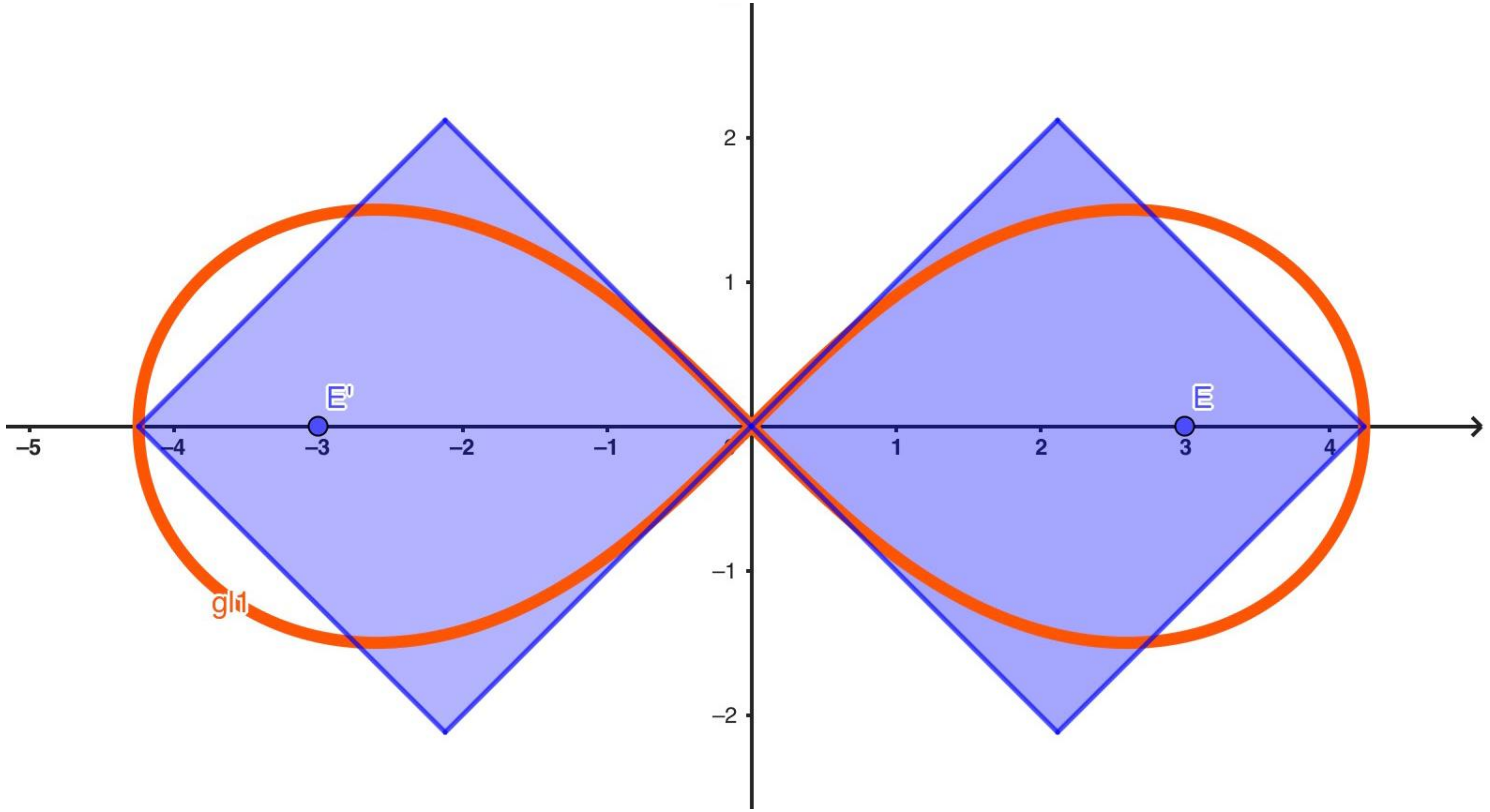
$$S = (\pm\frac{1}{2}\sqrt{2}e\,, \pm\frac{1}{2}\sqrt{2}e)$$

Flächeninhalt

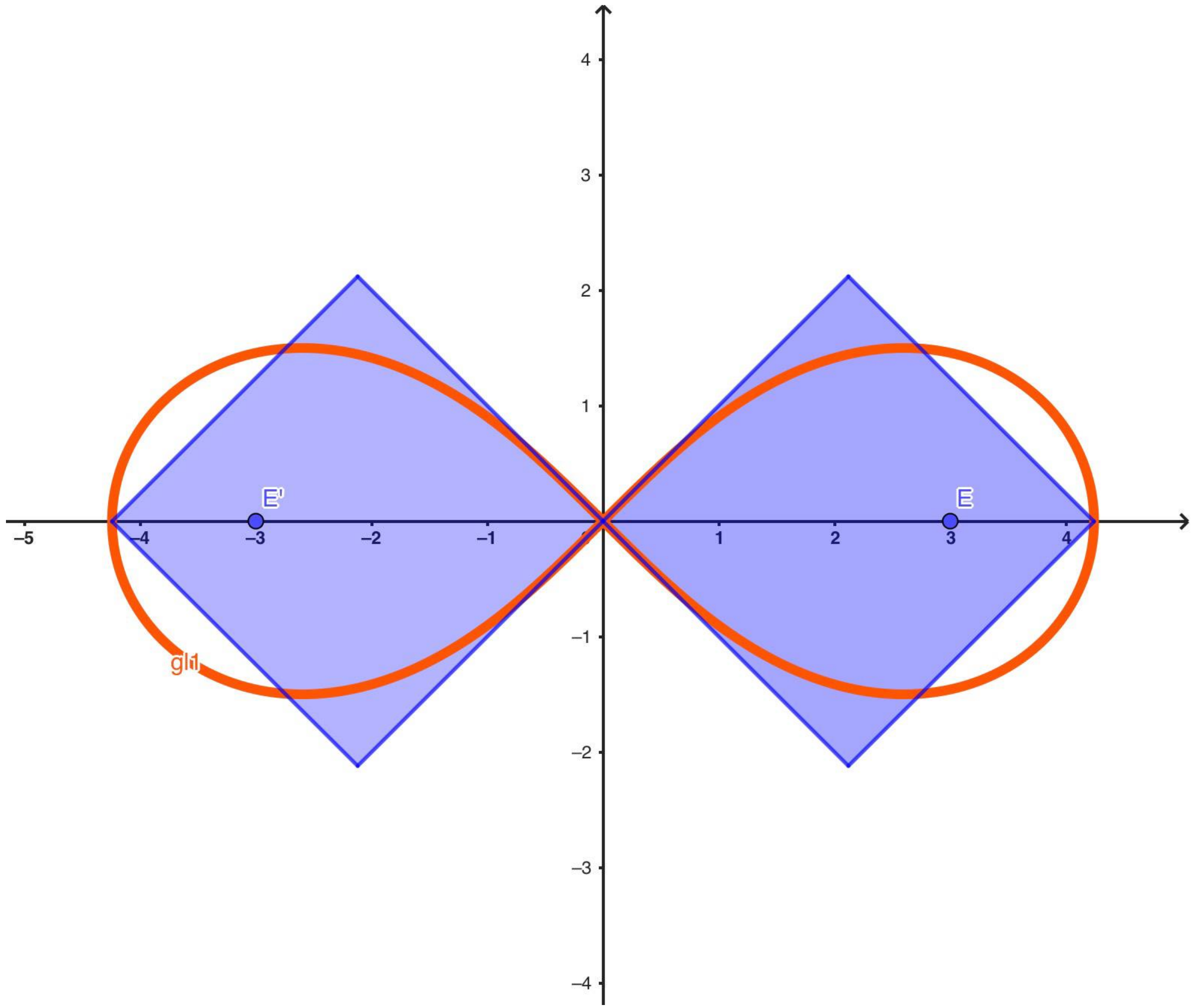
$$A(\varphi_0, \varphi_1) = \frac{1}{2} \int_{\varphi 0}^{\varphi 1} r(\varphi)^2 d\varphi$$

Flächeninhalt

$$A = 4 \cdot \frac{1}{2} \int_0^{\frac{\pi}{4}} (2e^2 \cos(2\varphi)) d(\varphi)$$



Flächeninhalt

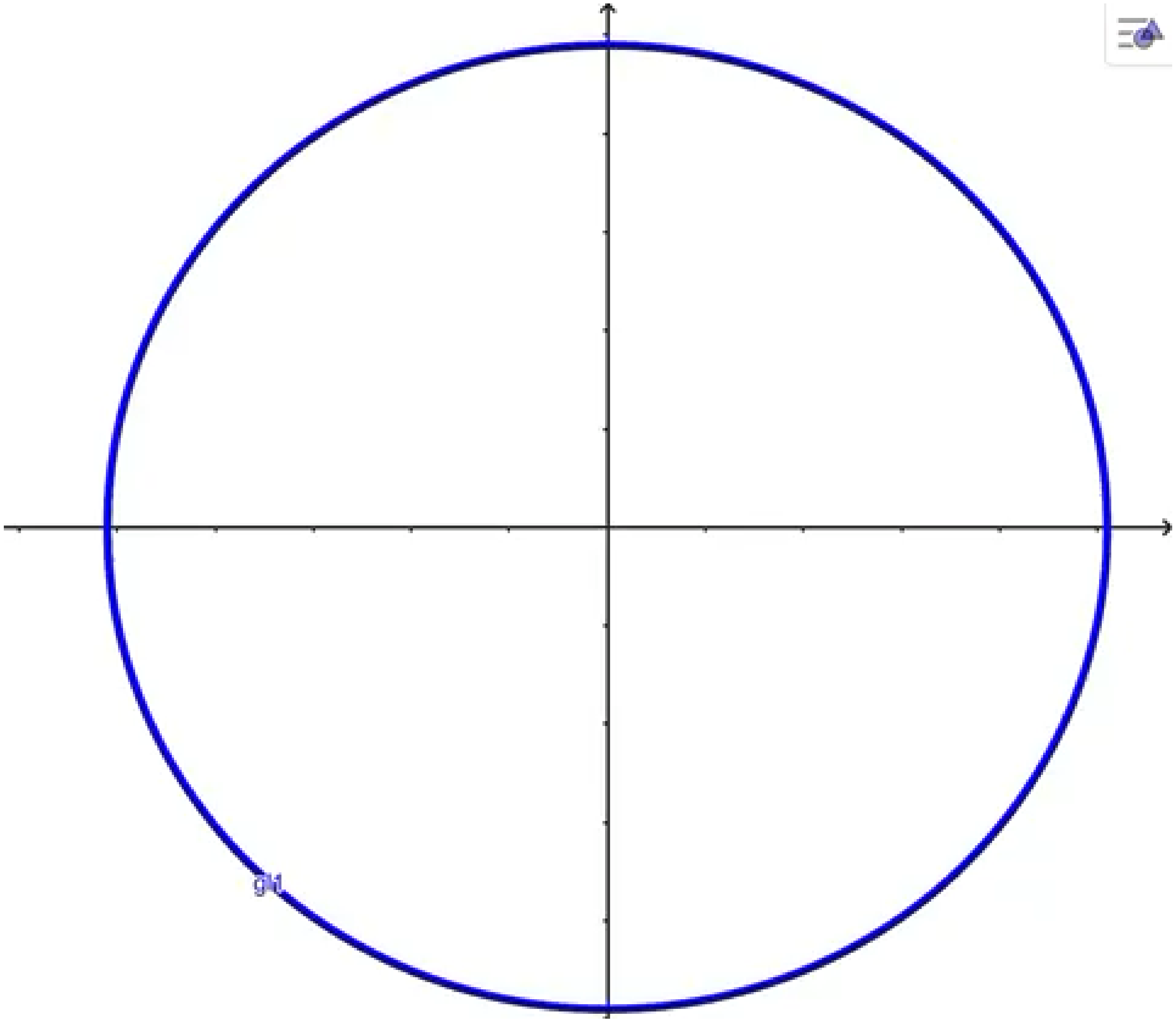


$$A =$$

$$4e^2\left[\frac{1}{2}\sin(2\varphi)\right]_0^{\frac{\pi}{4}}$$

$$A = 2e^2$$

Untersuchung der Cassini’schen Kurve


g1

Gleichungen

Kartesische Gleichung

$$\left((e-x)^2+y^2\right)\left((e+x)^2+y^2\right)=k^4$$

Implizite kartesische Gleichung

$$(x^2+y^2)^2-2e^2\,(x^2-y^2)=k^4-e^4$$

Nullstellen

$$\left((e-x)^2+0^2\right)\left((e+x)^2+0^2\right)=k^4$$

$$(e-x)^2\,(e+x)^2=k^4$$

$$\left(e^2-x^2\right)^2=k^4$$

$$x^2=e^2\pm k^2$$

$$x=\pm\sqrt{e^2\pm k^2}$$

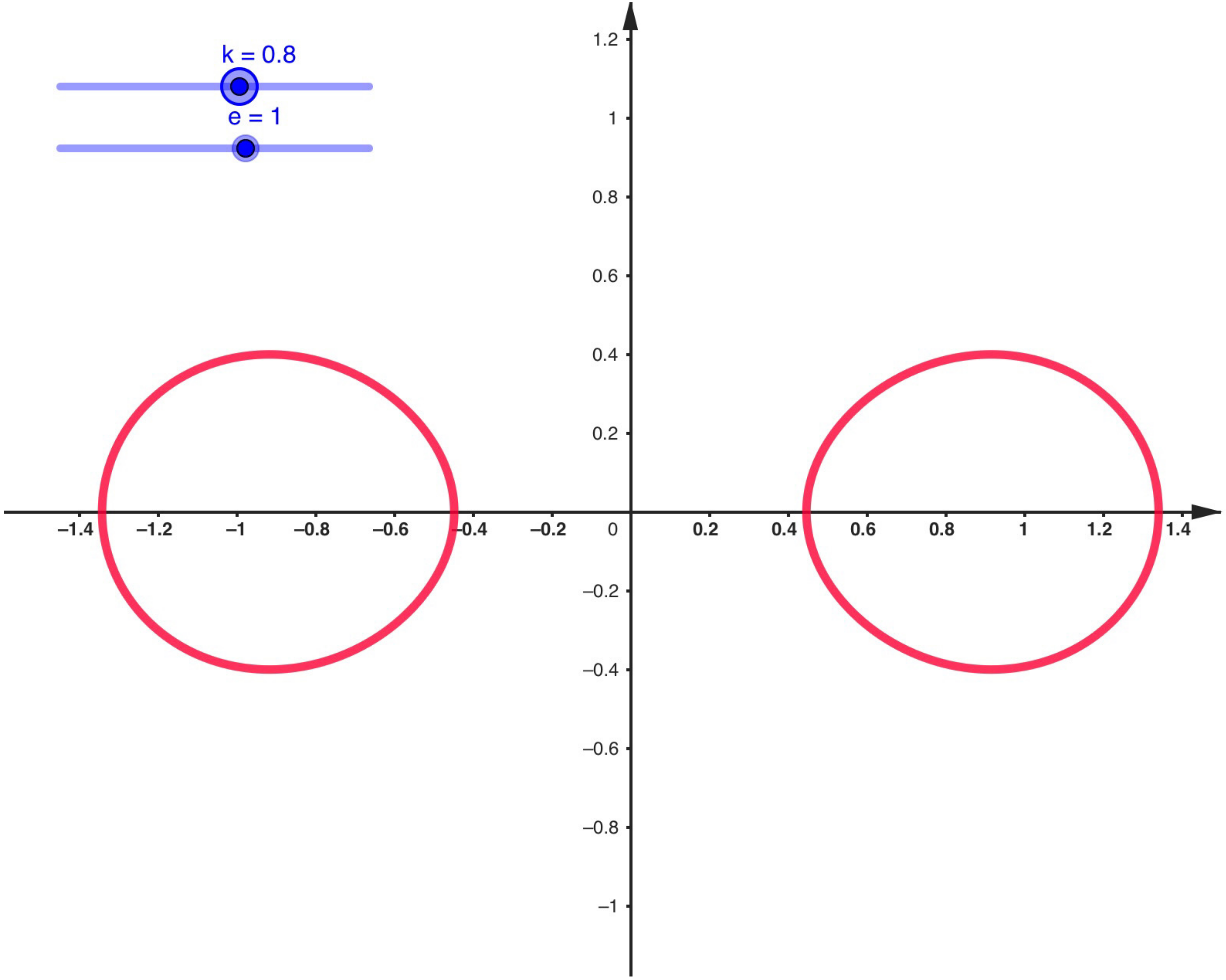
Schnittpunkte mit der y-Achse

$$\left((e-0)^2+y^2\right)\left((e+0)^2+y^2\right)=k^4$$

$$(e^2+y^2)^2=k^4$$

$$(e^2+y^2)=\pm k^2$$

$$y=\pm\sqrt{k^2-e^2}$$



$$\boldsymbol{k < e}$$

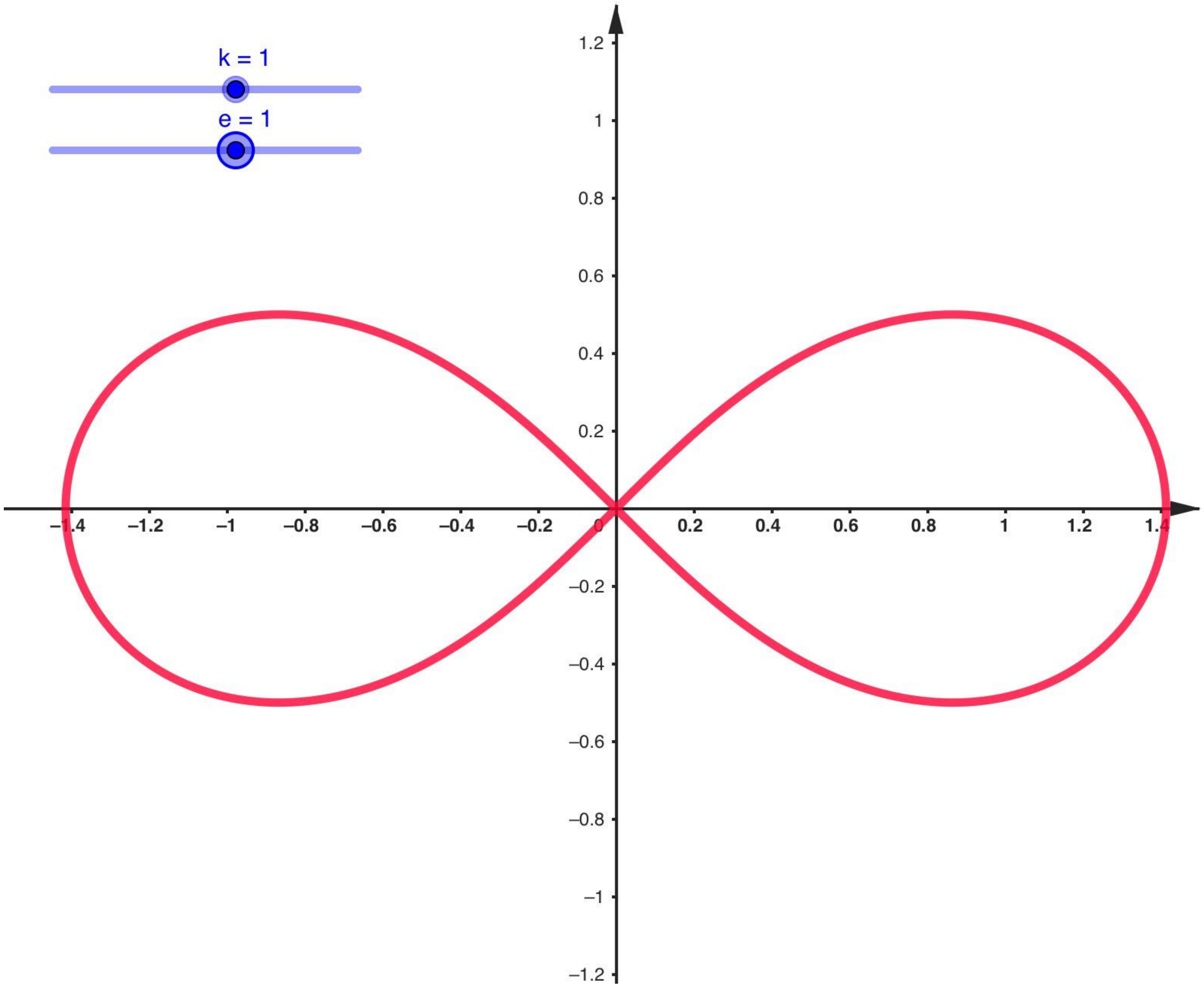
$$x = \pm\sqrt{e^2 \pm k^2}$$

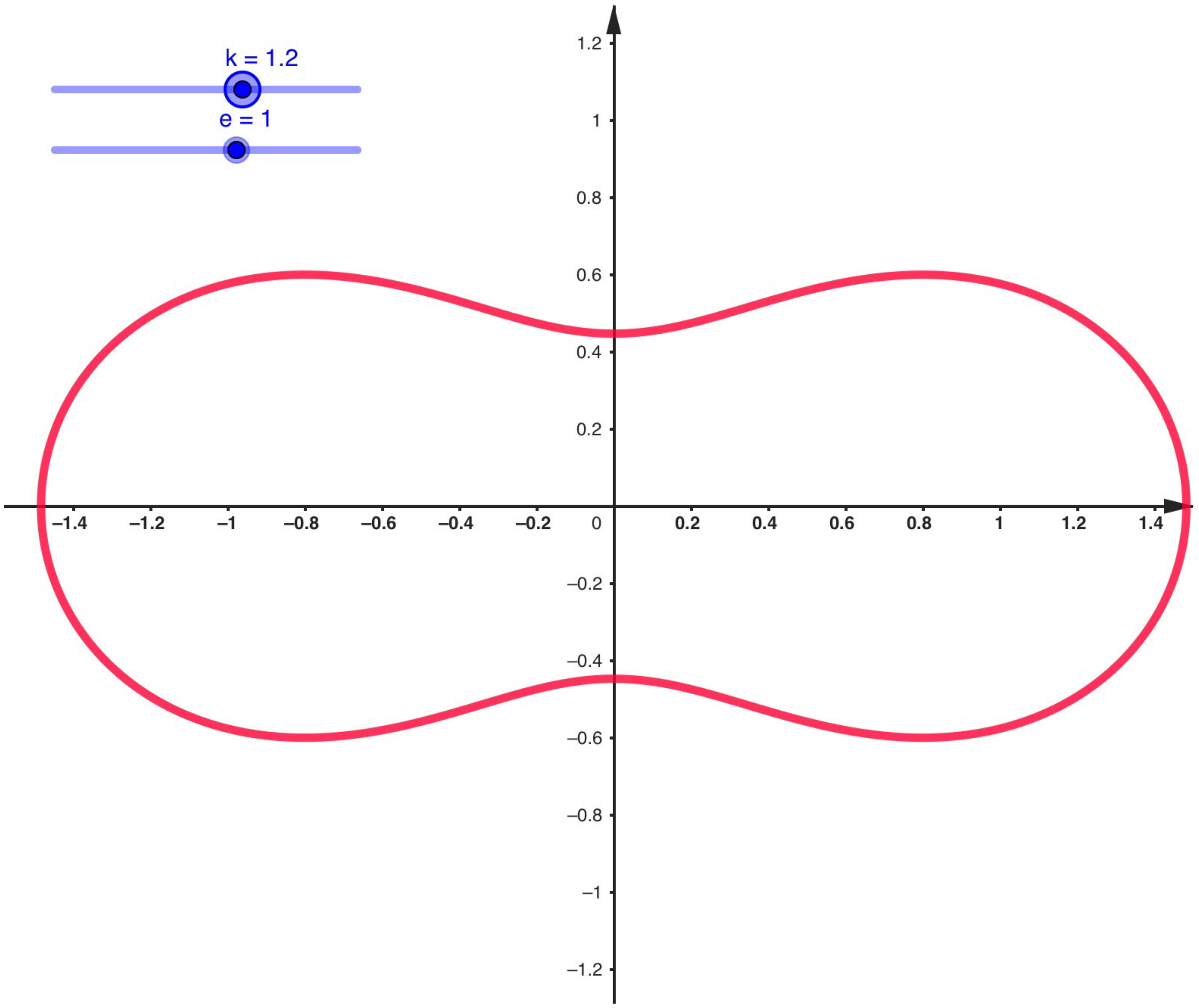
-> kein Schnittpunkt mit der y-Achse

$$\boldsymbol{k = e}$$

$$x = \pm\sqrt{2}e$$

$$y = 0$$





$$\boldsymbol{k > e}$$

$$x = \pm\sqrt{e^2 + k^2}$$

$$y = \pm\sqrt{k^2 - e^2}$$

Höhere Kurven im Unterricht

Pro:

- mathematikhistorische Bedeutung
- Realitätsbezug
- ästhetischer Reiz
- Entwicklung wichtiger Fähigkeiten (logisches und analytisches Denken, Problemlösungskompetenz)

Unterrichtliche Realisierung

- geometrische und trigonometrische Vorkenntnisse notwendig

➢ Sekundarstufe 2

- als Problemeinstieg in die Analysis

- fächerübergreifend